Basic Questions of
Bifurcations: Theory and Applications

1. Consider two smooth vector fields $f, g \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$ and a matrix $A \in \mathbb{R}^{N\times N}$. How is the Lie derivative of f in direction g defined? How is the Lie bracket of f and g defined? What is the adjoint operator $\mathrm{ad}A$?

2. Formulate the theorem on the normal form of a smooth vector field near an equilibrium via transposed of the linearization.

3. Given a smooth vector field f with linearization A at an equilibrium, assume that $\mathrm{ad}(A^T)$ has trivial kernel $\{0\}$ when restricted to the space of homogeneous polynomials of any degree $m \geq 2$. What is the normal form of f at the equilibrium?

4. Consider a smooth vector field f with linearization A at an equilibrium. Assume $AA^T = A^T A$. Which additional symmetry does the $\mathrm{ad}(A^T)$ normal form of f possess, to any finite order?

5. Consider a smooth vector field f with real diagonal linearization A at an equilibrium. When do resonances occur? How are they related to the normal form of f?

6. What is the normal form for Hopf bifurcation in the plane?

7. What is the Takens-Bogdanov-Arnol'd bifurcation in the plane? Describe some of its most relevant qualitative characteristics.

8. Formulate the theorem on the normal form of a smooth local diffeomorphism $\phi(x) = Bx + \ldots$ near the fixed point $x = 0$, via $B[\cdot]_m$.

9. What is a Lie group?

10. What is a Lie algebra?

11. How do Lie groups and Lie algebras relate to one another?

12. How does the $\mathrm{ad}(A^T)$ normal form of a vector field $f(x) = Ax + \ldots$ relate to the $B[\cdot]$ normal form of its time-1 map $\Phi_1(x) = Bx + \ldots$?

13. What is a Fredholm operator? What is the Fredholm index?

14. Let $L, M : X \to Y$ be bounded linear operators between Banach spaces. Which of the following are true? Which of them are false? How are Fredholm indices related, for the true statements?

   (a) Fredholm operators $X \to Y$ form a vector space.

   (b) The set of Fredholm operators $X \to Y$ is open.

   (c) If $||L||$ is small, then L is Fredholm.

   (d) If L is compact, then L is Fredholm.

   (e) If L is Fredholm and M is compact, then $L + M$ is Fredholm.

   (f) If $\dim X < \infty$ and $\dim Y < \infty$, then L is Fredholm.

   (g) Id_X is a Fredholm operator.

15. Describe the procedure of Lyapunov-Schmidt reduction for solving an equation $F(x) = 0$.

16. Formulate the finite-dimensional theorem by Crandall and Rabinowitz on stationary bifurcation from simple eigenvalues.

17. Give an explicit example of a pitchfork bifurcation and sketch the possible local bifurcation diagrams.

18. Give an explicit example of a saddle-node bifurcation and sketch the possible local bifurcation diagrams.

19. Formulate the buckling problem of the Euler beam

$$\ddot{x}(t) + \lambda \sin(x(t)) = 0, \qquad \dot{x}(0) = \dot{x}(1) = 0,$$

   as a stationary bifurcation problem, in the setting of Crandall and Rabinowitz.

20. What is a representation of a group Γ on a Banach space X? Give an example.

21. What is a strongly continuous representation of a topological group Γ on a Banach space X? How does it compare to a norm continuous representation?

22. Let Γ be a group with subgroup K and let ρ be a representation of Γ on a Banach space X. Define the set of K-fixed vectors. Define the isotropy Γ_x of an element $x \in X$?

23. Let Γ be a group with representation ρ on a Banach space X. How are the isotropy groups of elements on the same group orbit Γx related to each other? What is an isotropy subgroup of Γ?

24. When do we say that a map $F : X \to X$ is Γ-equivariant? When do we say that a subspace $V \leq X$ is Γ-invariant?

25. When do we say that a group representation is irreducible?

26. Formulate Schur's lemma. When is a representation of a group called absolutely irreducible?

27. Consider an irreducible complex representation of an Abelian group on a finite-dimensional complex vector space $X \neq \{0\}$. What is the complex dimension of X?

28. What are the irreducible representations of $SO(2)$ acting on $L^2(S^1)$ via the usual shift representation?

29. Under which conditions is a Lyapunov-Schmidt reduced equation equivariant under a group Γ?

30. Formulate Vanderbauwhede's equivariant branching lemma. Sketch a proof. (You may assume that results seen previously during the course hold.)

31. Formulate the Hopf bifurcation theorem and sketch a possible local bifurcation diagram.

32. Formulate the bifurcation of periodic orbits in the Hopf setting, as a stationary $SO(2)$-equivariant bifurcation problem in suitable Banach spaces.

33. What is a reversible vector field? Give an example.

34. How does reversibility affect the Hopf bifurcation scenario? What is the equivariance group of the resulting stationary bifurcation problem in suitable Banach spaces of 2π-periodic functions?

35. Formulate the theorem on reversible Hopf bifurcation.

36. Formulate the theorem on local bifurcation of subharmonic solutions.

37. Given a periodic solution of an ODE, what is its (spatio-temporal) symmetry?

38. Formulate the theorem on equivariant Hopf bifurcation.